A

SHORT TABLE OF INTEGRALS.

COMPILED BY

B. O. PEIRCE,

HOLLIS PROFESSOR OF MATHEMATICS AND NATURAL PHILOSOPHY
IN HARVARD UNIVERSITY.

BOSTON, U.S.A.:
GINN & COMPANY, PUBLISHERS.
1889.

The compiler will be grateful to any person who may send notice of errors in these formulas to

B. O. PEIRCE,

Harvard College, Cambridge.

I. FUNDAMENTAL FORMS.

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1. $\int a\,dx = ax.$

2. $\int af(x)\,dx = a\int f(x)\,dx.$

3. $\int \frac{dx}{x} = \log x.$

4. $\int x^m dx = \frac{x^{m+1}}{m+1}$, when m is different from -1.

5. $\int e^x dx = e^x.$

6. $\int a^x \log a\,dx = a^x.$

7. $\int \frac{dx}{1+x^2} = \tan^{-1}x.$

8. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x.$

9. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x.$

10. $\int \frac{dx}{\sqrt{2x-x^2}} = \text{versin}^{-1}x.$

11. $\int \cos x\,dx = \sin x.$

12. $\int \sin x\,dx = -\cos x.$

13. $\int \operatorname{ctn} x\,dx = \log \sin x.$

14. $\int \tan x\,dx = -\log \cos x.$

15. $\int \tan x \sec x\,dx = \sec x.$

16. $\int \sec^2 x\,dx = \tan x.$

17. $\int \csc^2 x\,dx = -\operatorname{ctn} x.$

In the following formulas, u, v, w, and y represent any functions of x:

18. $\int (u+v+w+\text{etc.})\,dx = \int u dx + \int v dx + \int w dx + \text{etc.}$

19a. $\int u dv = uv - \int v du.$

19b. $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx.$

20. $\int f(y)\,dx = \int \frac{f(y)\,dy}{\frac{dy}{dx}}.$

$\int a^x dx = \frac{a^x}{\log a}$

II. RATIONAL ALGEBRAIC FUNCTIONS.

A. — Expressions Involving $(a + bx)$.

The substitution of y or z for x, where $y = xz = a + bx$, gives

21. $$\int (a+bx)^m dx = \frac{1}{b}\int y^m dy.$$

22. $$\int x(a+bx)^m dx = \frac{1}{b^2}\int y^m (y-a) dy.$$

23. $$\int x^n (a+bx)^m dx = \frac{1}{b^{n+1}}\int y^m (y-a)^n dy.$$

24. $$\int \frac{x^n dx}{(a+bx)^m} = \frac{1}{b^{n+1}}\int \frac{(y-a)^n dy}{y^m}.$$

25. $$\int \frac{dx}{x^n (a+bx)^m} = -\frac{1}{a^{m+n-1}}\int \frac{(z-b)^{m+n-2} dz}{z^m}.$$

Whence

26. $$\int \frac{dx}{a+bx} = \frac{1}{b}\log(a+bx).$$

27. $$\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}.$$

28. $$\int \frac{dx}{(a+bx)^3} = -\frac{1}{2b(a+bx)^2}.$$

29. $$\int \frac{x\,dx}{a+bx} = \frac{1}{b^2}[a+bx-a\log(a+bx)].$$

30. $$\int \frac{x\,dx}{(a+bx)^2} = \frac{1}{b^2}\left[\log(a+bx)+\frac{a}{a+bx}\right]$$

31. $\int \frac{x\,dx}{(a+bx)^3} = \frac{1}{b^2}\left[-\frac{1}{a+bx} + \frac{a}{2(a+bx)^2}\right]$.

32. $\int \frac{x^2\,dx}{a+bx} = \frac{1}{b^3}\left[\tfrac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\log(a+bx)\right]$.

33. $\int \frac{x^2\,dx}{(a+bx)^2} = \frac{1}{b^3}\left[a+bx-2a\log(a+bx) - \frac{a^2}{a+bx}\right]$.

34. $\int \frac{dx}{x(a+bx)} = -\frac{1}{a}\log\frac{a+bx}{x}$.

35. $\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\log\frac{a+bx}{x}$.

36. $\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\log\frac{a+bx}{x}$.

B. — Expressions Involving $(a+bx^n)$.

37. $\int \frac{dx}{c^2+x^2} = \frac{1}{c}\tan^{-1}\frac{x}{c}$.

38. $\int \frac{dx}{c^2-x^2} = \frac{1}{2c}\log\frac{c+x}{c-x}$.

39. $\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}}\tan^{-1}x\sqrt{\frac{b}{a}}$, if $a>0,\ b>0$.

40. $\int \frac{dx}{a+bx^2} = \frac{1}{2\sqrt{-ab}}\log\frac{\sqrt{a}+x\sqrt{-b}}{\sqrt{a}-x\sqrt{-b}}$, if $a>0,\ b<0$.

41. $\int \frac{dx}{(a+bx^2)^2} = \frac{x}{2a(a+bx^2)} + \frac{1}{2a}\int\frac{dx}{a+bx^2}$.

42. $\int \frac{dx}{(a+bx^2)^{m+1}} = \frac{1}{2ma}\,\frac{x}{(a+bx^2)^m} + \frac{2m-1}{2ma}\int\frac{dx}{(a+bx^2)^m}$.

43. $\int \frac{x\,dx}{a+bx^2} = \frac{1}{2b}\log\left(x^2+\frac{a}{b}\right)$.

44. $\int \frac{x\,dx}{(a+bx^2)^{m+1}} = \frac{1}{2}\int \frac{dz}{(a+bz)^{m+1}}$, where $z = x^2$.

45. $\int \frac{dx}{x(a+bx^2)} = \frac{1}{2a}\log\frac{x^2}{a+bx^2}$.

46. $\int \frac{x^2\,dx}{a+bx^2} = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a+bx^2}$.

47. $\int \frac{dx}{x^2(a+bx^2)} = -\frac{1}{ax} - \frac{b}{a}\int \frac{dx}{a+bx^2}$.

48. $\int \frac{x^2\,dx}{(a+bx^2)^{m+1}} = \frac{-x}{2mb(a+bx^2)^m} + \frac{1}{2mb}\int \frac{dx}{(a+bx^2)^m}$.

49. $\int \frac{dx}{x^2(a+bx^2)^{m+1}} = \frac{1}{a}\int \frac{dx}{x^2(a+bx^2)^m} - \frac{b}{a}\int \frac{dx}{(a+bx^2)^{m+1}}$.

50. $\int \frac{dx}{a+bx^3} = \frac{k}{3a}\left[\frac{1}{2}\log\left(\frac{(k+x)^2}{k^2-kx+x^2}\right) + \sqrt{3}\tan^{-1}\frac{2x-k}{k\sqrt{3}}\right]$,

where $bk^3 = a$.

51. $\int \frac{x\,dx}{a+bx^3} = \frac{1}{3bk}\left[\frac{1}{2}\log\left(\frac{k^2-kx+x^2}{(k+x)^2}\right) + \sqrt{3}\tan^{-1}\frac{2x-k}{k\sqrt{3}}\right]$,

where $bk^3 = a$.

52. $\int \frac{dx}{x(a+bx^n)} = \frac{1}{an}\log\frac{x^n}{a+bx^n}$.

53. $\int \frac{dx}{(a+bx^n)^{m+1}} = \frac{1}{a}\int \frac{dx}{(a+bx^n)^m} - \frac{b}{a}\int \frac{x^n\,dx}{(a+bx^n)^{m+1}}$.

54. $\int \frac{x^m\,dx}{(a+bx^n)^{p+1}} = \frac{1}{b}\int \frac{x^{m-n}}{(a+bx^n)^p} - \frac{a}{b}\int \frac{x^{m-n}\,dx}{(a+bx^n)^{p+1}}$.

55. $\int \frac{dx}{x^m(a+bx^n)^{p+1}} = \frac{1}{a}\int \frac{dx}{x^m(a+bx^n)^p} - \frac{b}{a}\int \frac{dx}{x^{m-n}(a+bx^n)^{p+1}}$.

56. $$\int x^{m-1}(a+bx^n)^p\,dx = \begin{cases} \dfrac{1}{b(m+np)}\left[x^{m-n}(a+bx^n)^{p+1} - (m-n)a\int x^{m-n-1}(a+bx^n)^p\,dx\right]. \\ \dfrac{1}{m+np}\left[x^m(a+bx^n)^p + npa\int x^{m-1}(a+bx^n)^{p-1}dx\right]. \\ \dfrac{1}{ma}\left[x^m(a+bx^n)^{p+1} - (m+np+n)b\int x^{m+n-1}(a+bx^n)^p\,dx\right]. \\ \dfrac{1}{an(p+1)}\left[-x^m(a+bx^n)^{p+1} + (m+np+n)\int x^{m-1}(a+bx^n)^{p+1}dx\right]. \end{cases}$$

C. — Expressions Involving $(a+bx+cx^2)$.

Let $X = a+bx+cx^2$ and $q = 4ac-b^2$, then

57. $$\int\frac{dx}{X} = \frac{2}{\sqrt{q}}\tan^{-1}\frac{2cx+b}{\sqrt{q}}, \text{ when } q>0.$$

58. $$\int\frac{dx}{X} = \frac{1}{\sqrt{-q}}\log\frac{2cx+b-\sqrt{-q}}{2cx+b+\sqrt{-q}}, \text{ when } q<0.$$

59. $$\int\frac{dx}{X^2} = \frac{2cx+b}{qX} + \frac{2c}{q}\int\frac{dx}{X}.$$

60. $$\int\frac{dx}{X^3} = \frac{2cx+b}{q}\left(\frac{1}{2X^2} + \frac{3c}{qX}\right) + \frac{6c^2}{q^2}\int\frac{dx}{X}.$$

61. $$\int\frac{dx}{X^{n+1}} = \frac{2cx+b}{nqX^n} + \frac{2(2n-1)c}{qn}\int\frac{dx}{X^n}.$$

62. $$\int\frac{x\,dx}{X} = \frac{1}{2c}\log X - \frac{b}{2c}\int\frac{dx}{X}.$$

63. $$\int \frac{x\,dx}{X^2} = -\frac{bx+2a}{qX} - \frac{b}{q}\int \frac{dx}{X}.$$

64. $$\int \frac{x\,dx}{X^{n+1}} = -\frac{2a+bx}{nqX^n} - \frac{b(2n-1)}{nq}\int \frac{dx}{X^n}.$$

65. $$\int \frac{x^2}{X}dx = \frac{x}{c} - \frac{b}{2c^2}\log X + \frac{b^2-2ac}{2c^2}\int \frac{dx}{X}.$$

66. $$\int \frac{x^2}{X^2}dx = \frac{(b^2-2ac)x+ab}{cqX} + \frac{2a}{q}\int \frac{dx}{X}.$$

67. $$\int \frac{x^m dx}{X^{n+1}} = -\frac{x^{m-1}}{(2n-m+1)cX^n} - \frac{n-m+1}{2n-m+1}\cdot\frac{b}{c}\int \frac{x^{m-1}dx}{X^{n+1}}$$
$$+\frac{m-1}{2n-m+1}\cdot\frac{a}{c}\int \frac{x^{m-2}dx}{X^{n+1}}.$$

68. $$\int \frac{dx}{xX} = \frac{1}{2a}\log\frac{x^2}{X} - \frac{b}{2a}\int \frac{dx}{X}.$$

69. $$\int \frac{dx}{x^2X} = \frac{b}{2a^2}\log\frac{X}{x^2} - \frac{1}{ax} + \left(\frac{b^2}{2a^2} - \frac{c}{a}\right)\int \frac{dx}{X}.$$

70. $$\int \frac{dx}{x^m X^{n+1}} = -\frac{1}{(m-1)ax^{m-1}X^n} - \frac{n+m-1}{m-1}\cdot\frac{b}{a}\int \frac{dx}{x^{m-1}X^{n+1}}$$
$$-\frac{2n+m-1}{m-1}\cdot\frac{c}{a}\int \frac{dx}{x^{m-2}X^{n+1}}.$$

D. — Rational Fractions.

Every proper fraction can be represented by the general form:

$$\frac{f(x)}{F(x)} = \frac{g_1x^{n-1}+g_2x^{n-2}+g_3x^{n-3}+\cdots+g_n}{x^n+k_1x^{n-1}+k_2x^{n-2}+\cdots+k_n}.$$

a, b, c, etc., are the roots of the equation $F(x)=0$, so that

$$F(x) = (x-a)^p(x-b)^q(x-c)^r\cdots,$$

then
$$\frac{f(x)}{F(x)} = \frac{A_1}{(x-a)^p} + \frac{A_2}{(x-a)^{p-1}} + \frac{A_3}{(x-a)^{p-2}} + \cdots + \frac{A_p}{x-a}$$
$$+ \frac{B_1}{(x-b)^q} + \frac{B_2}{(x-b)^{q-1}} + \frac{B_3}{(x-b)^{q-2}} + \cdots + \frac{B_q}{x-b}$$
$$+ \frac{C_1}{(x-c)^r} + \frac{C_2}{(x-c)^{r-1}} + \frac{C_3}{(x-c)^{r-2}} + \cdots + \frac{C_r}{x-c}$$
$$+ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

Where the numerators of the separate fractions may be determined by the equations

$$A_m = \frac{\phi_1^{[m-1]}(a)}{(m-1)!}, \quad B_m = \frac{\phi_2^{[m-1]}(b)}{(m-1)!}, \text{ etc., etc.}$$

$$\phi_1(x) = \frac{f(x)(x-a)^p}{F(x)}, \quad \phi_2(x) = \frac{f(x)(x-b)^q}{F(x)}, \text{ etc., etc.}$$

If a, b, c, etc., are single roots, then $p = q = r = \cdots = 1$, and
$$\frac{f(x)}{F(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \cdots$$

where $A = \frac{f(a)}{F'(a)}$, $B = \frac{f(b)}{F'(b)}$, etc.

The simpler fractions, into which the original fraction is thus divided, may be integrated by means of the following formulas:

71. $$\int \frac{h\,dx}{(mx+n)^l} = \int \frac{h\,d(mx+n)}{m(mx+n)^l} = \frac{h}{m(1-l)(mx+n)^{l-1}}.$$

72. $$\int \frac{h\,dx}{mx+n} = \frac{h}{m} \log(mx+n).$$

If any of the roots of the equation $f(x) = 0$ are imaginary, the parts of the integral which arise from conjugate roots can be combined together and the integral brought into a real form. The following formula, in which $i = \sqrt{-1}$, is often useful in combining logarithms of conjugate complex quantities:

78. $$\log(x \pm yi) = \tfrac{1}{2}\log(x^2+y^2) \pm i \tan^{-1}\frac{y}{x}.$$

III. IRRATIONAL ALGEBRAIC FUNCTIONS.

A. — Expressions Involving $\sqrt{a+bx}$.

The substitution of a new variable of integration, $y = \sqrt{a+bx}$, gives

74. $$\int \sqrt{a+bx}\, dx = \frac{2}{3b}\sqrt{(a+bx)^3}.$$

75. $$\int x\sqrt{a+bx}\, dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2}.$$

76. $$\int x^2\sqrt{a+bx}\, dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3}.$$

77. $$\int \frac{\sqrt{a+bx}}{x}dx = 2\sqrt{a+bx} + a\int \frac{dx}{x\sqrt{a+bx}}.$$

78. $$\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}.$$

79. $$\int \frac{x\,dx}{\sqrt{a+bx}} = -\frac{2(2a-bx)}{3b^2}\sqrt{a+bx}.$$

80. $$\int \frac{x^2\,dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx}.$$

81. $$\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}}\log\left(\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right), \text{ for } a>0.$$

82. $$\int \frac{dx}{x\sqrt{a+bx}} = \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bx}{-a}}, \text{ for } a<0.$$

83. $$\int \frac{dx}{x^2\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{a+bx}}.$$

84. $\int (a+bx)^{\pm\frac{n}{2}}\,dx = \frac{2}{b}\int y^{1\pm n}\,dy = \frac{2\,(a+bx)^{\frac{2\pm n}{2}}}{b\,(2\pm n)}.$

85. $\int x\,(a+bx)^{\pm\frac{n}{2}}\,dx = \frac{2}{b^2}\left[\frac{(a+bx)^{\frac{4\pm n}{2}}}{4\pm n} - \frac{a\,(a+bx)^{\frac{2\pm n}{2}}}{2\pm n}\right].$

86. $\int \frac{x^m\,dx}{\sqrt{a+bx}} = \frac{2\,x^m\sqrt{a+bx}}{(2m+1)b} - \frac{2ma}{(2m+1)b}\int \frac{x^{m-1}\,dx}{\sqrt{a+bx}}.$

87. $\int \frac{dx}{x^n\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{(n-1)\,ax^{n-1}} - \frac{(2n-3)\,b}{(2n-2)\,a}\int \frac{dx}{x^{n-1}\sqrt{a+bx}}.$

88. $\int \frac{(a+bx)^{\frac{n}{2}}\,dx}{x} = b\int (a+bx)^{\frac{n-2}{2}}\,dx + a\int \frac{(a+bx)^{\frac{n-2}{2}}}{x}\,dx.$

89. $\int \frac{dx}{x\,(a+bx)^{\frac{m}{2}}} = \frac{1}{a}\int \frac{dx}{x\,(a+bx)^{\frac{m-2}{2}}} - \frac{b}{a}\int \frac{dx}{(a+bx)^{\frac{m}{2}}}.$

89(b.) $\int \frac{\sqrt{a+bx}}{x}\,dx =$

B. — Expressions* Involving $\sqrt{x^2 \pm a^2}$ and $\sqrt{a^2 - x^2}$.

90. $\int \sqrt{x^2 \pm a^2}\,dx = \tfrac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2 \log\left(x + \sqrt{x^2 \pm a^2}\right)\right].$

91. $\int \sqrt{a^2 - x^2}\,dx = \tfrac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right).$

92. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log\left(x + \sqrt{x^2 \pm a^2}\right).$

93. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}.$

94. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\cos^{-1}\frac{a}{x}.$

95. $\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = -\frac{1}{a}\log\left(\frac{a + \sqrt{a^2 \pm x^2}}{x}\right).$

96. $\int \frac{\sqrt{a^2 \pm x^2}}{x}\,dx = \sqrt{a^2 \pm x^2} - a\log\frac{a + \sqrt{a^2 \pm x^2}}{x}.$

*These equations are all special cases of more general equations given in the next section.

97. $\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\cos^{-1}\frac{a}{x}.$

98. $\int \frac{x\,dx}{\sqrt{a^2 \pm x^2}} = \pm\sqrt{a^2 \pm x^2}.$

99. $\int \frac{x\,dx}{\sqrt{x^2-a^2}} = \sqrt{x^2-a^2}.$

100. $\int x\sqrt{x^2 \pm a^2}\,dx = \tfrac{1}{3}\sqrt{(x^2 \pm a^2)^3}.$

101. $\int x\sqrt{a^2-x^2}\,dx = -\tfrac{1}{3}\sqrt{(a^2-x^2)^3}.$

102. $\int \sqrt{(x^2 \pm a^2)^3}\,dx$

$$= \tfrac{1}{4}\left[x\sqrt{(x^2 \pm a^2)^3} \pm \frac{3a^2x}{2}\sqrt{x^2 \pm a^2} + \frac{3a^4}{2}\log(x + \sqrt{x^2 \pm a^2})\right].$$

103. $\int \sqrt{(a^2-x^2)^3}\,dx$

$$= \tfrac{1}{4}\left[x\sqrt{(a^2-x^2)^3} + \frac{3a^2x}{2}\sqrt{a^2-x^2} + \frac{3a^4}{2}\sin^{-1}\frac{x}{a}\right].$$

104. $\int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}.$

105. $\int \frac{dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{a^2\sqrt{a^2-x^2}}.$

106. $\int \frac{x\,dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-1}{\sqrt{x^2 \pm a^2}}.$

107. $\int \frac{x\,dx}{\sqrt{(a^2-x^2)^3}} = \frac{1}{\sqrt{a^2-x^2}}.$

108. $\int x\sqrt{(x^2 \pm a^2)^3}\,dx = \tfrac{1}{5}\sqrt{(x^2 \pm a^2)^5}.$

109. $\int x\sqrt{(a^2-x^2)^3}\,dx = -\tfrac{1}{5}\sqrt{(a^2-x^2)^5}.$

110. $\int x^2\sqrt{x^2 \pm a^2}\,dx$

$$= \frac{x}{4}\sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8}\left(x\sqrt{x^2 \pm a^2} \pm a^2 \log\left(x + \sqrt{x^2 \pm a^2}\right)\right).$$

111. $\int x^2\sqrt{a^2 - x^2}\,dx$

$$= -\frac{x}{4}\sqrt{(a^2 - x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right).$$

112. $\int \frac{x^2\,dx}{\sqrt{x^2 \pm a^2}} = \frac{x}{2}\sqrt{x^2 \pm a^2} \mp \frac{a^2}{2}\log\left(x + \sqrt{x^2 \pm a^2}\right).$

113. $\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

114. $\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \pm\frac{\sqrt{x^2 \pm a^2}}{a^2 x}.$

115. $\int \frac{dx}{x^2\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x}.$

116. $\int \frac{\sqrt{x^2 \pm a^2}\,dx}{x^2} = -\frac{\sqrt{x^2 \pm a^2}}{x} + \log\left(x + \sqrt{x^2 \pm a^2}\right).$

117. $\int \frac{\sqrt{a^2 - x^2}}{x^2}\,dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1}\frac{x}{a}.$

118. $\int \frac{x^2\,dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-x}{\sqrt{x^2 \pm a^2}} + \log\left(x + \sqrt{x^2 \pm a^2}\right).$

119. $\int \frac{x^2\,dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1}\frac{x}{a}.$

C. — Expressions Involving $\sqrt{a + bx + cx^2}$.

Let $X = a + bx + cx^2$, $q = 4ac - b^2$, and $k = \frac{4c}{q}$. In order to rationalize the function $f(x, \sqrt{a + bx + cx^2})$ we may put $\sqrt{a + bx + cx^2} = \sqrt{\pm c}\sqrt{A + Bx \pm x^2}$, according as c is positive or negative, and then substitute for x a new variable z, such that

$z = \sqrt{A + Bx + x^2} - x$, if $c > 0$.

$z = \dfrac{\sqrt{A + Bx - x^2} - \sqrt{A}}{x}$, if $c < 0$ and $\dfrac{a}{-c} > 0$.

$z = \sqrt{\dfrac{x - \beta}{a - x}}$, where a and β are the roots of the equation

$$A + Bx - x^2 = 0, \text{ if } c < 0 \text{ and } \frac{a}{-c} < 0.$$

By rationalization, or by the aid of reduction formulas, may be obtained the values of the following integrals:

120. $\displaystyle\int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{c}} \log\left(\sqrt{X} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right)$, if $c > 0$.

121. $\displaystyle\int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{-c}} \sin^{-1}\left(\frac{-2cx - b}{\sqrt{b^2 - 4ac}}\right)$, if $c < 0$.

122. $\displaystyle\int \frac{dx}{X\sqrt{X}} = \frac{2(2cx + b)}{q\sqrt{X}}$.

123. $\displaystyle\int \frac{dx}{X^2\sqrt{X}} = \frac{2(2cx + b)}{3q\sqrt{X}}\left(\frac{1}{X} + 2k\right)$.

124. $\displaystyle\int \frac{dx}{X^n\sqrt{X}} = \frac{2(2cx + b\sqrt{X}}{(2n-1)qX^n} + \frac{2k(n-1)}{2n-1}\int \frac{dx}{X^{n-1}\sqrt{X}}$.

125. $\displaystyle\int \sqrt{X}\,dx = \frac{(2cx + b)\sqrt{X}}{4c} + \frac{1}{2k}\int \frac{dx}{\sqrt{X}}$.

126. $\displaystyle\int X\sqrt{X}\,dx = \frac{(2cx + b)\sqrt{X}}{8c}\left(X + \frac{3}{2k}\right) + \frac{3}{8k^2}\int \frac{dx}{\sqrt{X}}$.

127. $\displaystyle\int X^2\sqrt{X}\,dx = \frac{(2cx+b)\sqrt{X}}{12c}\left(X^2 + \frac{5X}{4k} + \frac{15}{8k^2}\right) + \frac{5}{16k^3}\int \frac{dx}{\sqrt{X}}$.

128. $\displaystyle\int X^n\sqrt{X}\,dx = \frac{(2cx + b)X^n\sqrt{X}}{4(n+1)c} + \frac{2n+1}{2(n+1)k}\int \frac{X^n dx}{\sqrt{X}}$.

129. $\displaystyle\int \frac{x\,dx}{\sqrt{X}} = \frac{\sqrt{X}}{c} - \frac{b}{2c}\int \frac{dx}{\sqrt{X}}$.

180. $\int \frac{x\,dx}{X\sqrt{X}} = -\frac{2(bx+2a)}{q\sqrt{X}}.$

181. $\int \frac{x\,dx}{X^n\sqrt{X}} = -\frac{\sqrt{X}}{(2n-1)c\,X^n} - \frac{b}{2c}\int \frac{dx}{X^n\sqrt{X}}.$

182. $\int \frac{x^2\,dx}{\sqrt{X}} = \left(\frac{x}{2c} - \frac{3b}{4c^2}\right)\sqrt{X} + \frac{3b^2-4ac}{8c^2}\int \frac{dx}{\sqrt{X}}.$

183. $\int \frac{x^2\,dx}{X\sqrt{X}} = \frac{(2b^2-4ac)\,x+2ab}{cq\sqrt{X}} + \frac{1}{c}\int \frac{dx}{\sqrt{X}}.$

184. $\int \frac{x^2\,dx}{X^n\sqrt{X}} = \frac{(2b^2-4ac)\,x+2ab}{(2n-1)cqX^{n-1}\sqrt{X}} + \frac{4ac+(2n-3)b^2}{(2n-1)\,cq}\int \frac{dx}{X^{n-1}\sqrt{X}}.$

185. $\int \frac{x^3\,dx}{\sqrt{X}} = \left(\frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^2}{8c^3} - \frac{2a}{3c^2}\right)\sqrt{X} + \left(\frac{3ab}{4c^2} - \frac{5b^3}{16c^3}\right)\int \frac{dx}{\sqrt{X}}.$

186. $\int x\sqrt{X}\,dx = \frac{X\sqrt{X}}{3c} - \frac{b}{2c}\int \sqrt{X}\,dx.$

187. $\int xX\sqrt{X}\,dx = \frac{X^2\sqrt{X}}{5c} - \frac{b}{2c}\int X\sqrt{X}\,dx.$

188. $\int \frac{xX^n\,dx}{\sqrt{X}} = \frac{X^n\sqrt{X}}{(2n+1)c} - \frac{b}{2c}\int \frac{X^n\,dx}{\sqrt{X}}.$

189. $\int x^2\sqrt{X}\,dx = \left(x - \frac{5b}{6c}\right)\frac{X\sqrt{X}}{4c} + \frac{5b^2-4ac}{16c^2}\int \sqrt{X}\,dx.$

190. $\int \frac{x^2X^n\,dx}{\sqrt{X}} = \frac{xX^n\sqrt{X}}{2(n+1)c} - \frac{(2n+3)\,b}{4(n+1)\,c}\int \frac{xX^n\,dx}{\sqrt{X}} - \frac{a}{2(n+1)\,c}\int \frac{X^n\,dx}{\sqrt{X}}.$

191. $\int x^3\sqrt{X}\,dx = \left(x^3 - \frac{7bx}{8c} + \frac{35b^2}{48c^2} - \frac{2a}{3c}\right)\frac{X\sqrt{X}}{5c} + \left(\frac{3ab}{8c^2} - \frac{7b^3}{32c^3}\right)\int \sqrt{X}\,dx.$

142. $\int \frac{dx}{x\sqrt{X}} = -\frac{1}{\sqrt{a}} \log\left(\frac{\sqrt{X}+\sqrt{a}}{x} + \frac{b}{2\sqrt{a}}\right)$, if $a > 0$.

143. $\int \frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{bx+2a}{x\sqrt{b^2-4ac}}\right)$, if $a < 0$.

144. $\int \frac{dx}{x\sqrt{X}} = -\frac{2\sqrt{X}}{bx}$, if $a = 0$.

145. $\int \frac{dx}{xX^n\sqrt{X}} = \frac{\sqrt{X}}{(2n-1)aX^n} + \frac{1}{a}\int \frac{dx}{xX^{n-1}\sqrt{X}} - \frac{b}{2a}\int \frac{dx}{X^n\sqrt{X}}$.

146. $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{X}}$.

147. $\int \frac{\sqrt{X}\,dx}{x} = \sqrt{X} + \frac{b}{2}\int \frac{dx}{\sqrt{X}} + a\int \frac{dx}{x\sqrt{X}}$.

148. $\int \frac{X^n dx}{x\sqrt{X}} = \frac{X^n}{(2n-1)\sqrt{X}} + a\int \frac{X^{n-1}dx}{x\sqrt{X}} + \frac{b}{2}\int \frac{X^{n-1}dx}{\sqrt{X}}$.

149. $\int \frac{\sqrt{X}\,dx}{x^2} = -\frac{\sqrt{X}}{x} + \frac{b}{2}\int \frac{dx}{x\sqrt{X}} + c\int \frac{dx}{\sqrt{X}}$.

150. $\int \frac{x^m dx}{X^n\sqrt{X}} = \frac{1}{c}\int \frac{x^{m-2}dx}{X^{n-1}\sqrt{X}} - \frac{b}{c}\int \frac{x^{m-1}dx}{X^n\sqrt{X}} - \frac{a}{c}\int \frac{x^{m-2}dx}{X^n\sqrt{X}}$.

151. $\int \frac{x^m X^n dx}{\sqrt{X}} = \frac{x^{m-1}X^n\sqrt{X}}{(2n+m)c} - \frac{(2n+2m-1)b}{2c(2n+m)}\int \frac{x^{m-1}X^n dx}{\sqrt{X}}$
$- \frac{(m-1)a}{(2n+m)c}\int \frac{x^{m-2}X^n dx}{\sqrt{X}}$.

152. $\int \frac{dx}{x^m X^n\sqrt{X}} = -\frac{\sqrt{X}}{(m-1)ax^{m-1}X^n}$
$- \frac{(2n+2m-3)b}{2a(m-1)}\int \frac{dx}{x^{m-1}X^n\sqrt{X}} - \frac{(2n+m-2)c}{(m-1)a}\int \frac{dx}{x^{m-2}X^n\sqrt{X}}$.

158. $$\int \frac{X^n dx}{x^m \sqrt{X}} = -\frac{X^{n-1}\sqrt{X}}{(m-1)\,x^{m-1}} + \frac{(2n-1)\,b}{2\,(m-1)} \int \frac{X^{n-1} dx}{x^{m-1}\sqrt{X}} + \frac{(2n-1)\,c}{m-1} \int \frac{X^{n-1} dx}{x^{m-2}\sqrt{X}}.$$

D. — Miscellaneous Expressions.

154. $$\int \sqrt{2ax - x^2}\, dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2} \sin^{-1}\frac{x-a}{a}.$$

155. $$\int \frac{dx}{\sqrt{2ax - x^2}} = \operatorname{versin}^{-1}\frac{x}{a}.$$

156. $$\int \frac{dx}{(x+1)\sqrt{x^2-1}} = +\sqrt{\frac{x-1}{x+1}}.$$

157. $$\int \frac{dx}{(x-1)\sqrt{x^2-1}} = -\sqrt{\frac{x+1}{x-1}}.$$

158. $$\int \sqrt{\frac{1+x}{1-x}}\, dx = \sin^{-1}x - \sqrt{1-x^2}.$$

159. $$\int \sqrt{\frac{x+a}{x+b}}\, dx = \sqrt{(x+a)\,(x+b)} + (a-b)\log\left(\sqrt{x+a} + \sqrt{x+b}\right).$$

160. $$\int \frac{dx}{\sqrt{(x-\alpha)\,(\beta-x)}} = 2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}}.$$

161. $$\int \frac{dx}{\sqrt{(a+bx)\,(\alpha-\beta x)}} = \frac{2}{\sqrt{b\beta}} \sin^{-1}\sqrt{\frac{\beta\,(a+bx)}{a\beta+b\alpha}}.$$

162. $$\int \sqrt[3]{a+bx}\, dx = \frac{3}{4b}\sqrt[3]{(a+bx)^4}.$$

163. $$\int \frac{dx}{\sqrt[3]{a+bx}} = \frac{3}{2b}\sqrt[3]{(a+bx)^2}.$$

164. $\int \frac{x\,dx}{\sqrt[3]{a+bx}} = -\frac{3\,(3\,a-2\,bx)}{10\,b^2}\sqrt[3]{(a+bx)^2}.$

165. $\int \frac{dx}{x\sqrt{x^n-a^2}} = \frac{2}{an}\sec^{-1}\left(\frac{x^{\frac{n}{2}}}{a}\right).$

166. $\int \frac{dx}{x\sqrt{x^n+a^2}} = \frac{1}{an}\log\frac{\sqrt{a^2+x^n}-a}{\sqrt{a^2+x^n}+a}.$

IV. TRANSCENDENTAL FUNCTIONS.

167. $\int \sin x\,dx = -\cos x.$

168. $\int \sin^2 x\,dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x.$

169. $\int \sin^3 x\,dx = -\frac{1}{3}\cos x\,(\sin^2 x + 2).$

170. $\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$

171. $\int \cos x\,dx = \sin x.$

172. $\int \cos^2 x\,dx = \frac{1}{2}\sin x \cos x + \frac{1}{2}x.$

173. $\int \cos^3 x\,dx = \frac{1}{3}\sin x\,(\cos^2 x + 2).$

174. $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n\ 2} x\,dx.$

175. $\int \sin x \cos x\,dx = \frac{1}{2}\sin^2 x.$

176. $\int \sin^2 x \cos^2 x\,dx = -\frac{1}{8}\left(\frac{1}{4}\sin 4x - x\right).$

177. $\int \sin x \cos^m x\,dx = -\frac{\cos^{m+1} x}{m+1}.$

178. $\int \sin^m x \cos x\,dx = \frac{\sin^{m+1} x}{m+1}.$

179. $\int \cos^m x \sin^n x\,dx = \frac{\cos^{m-1} x \sin^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \cos^{m-2} x \sin^n x\,dx.$

180. $\int \cos^m x \sin^n x\,dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n}\int \cos^m x \sin^{n-2} x\,dx.$

181. $$\int \frac{\cos^m x\,dx}{\sin^n x} = -\frac{\cos^{m+1} x}{(n-1)\sin^{n-1} x} - \frac{m-n+2}{n-1}\int \frac{\cos^m x\,dx}{\sin^{n-2} x}.$$

182. $$\int \frac{\cos^m x\,dx}{\sin^n x} = \frac{\cos^{m-1} x}{(m-n)\sin^{n-1} x} + \frac{m-1}{m-n}\int \frac{\cos^{m-2} x\,dx}{\sin^n x}.$$

183. $$\int \frac{\sin^m x\,dx}{\cos^n x} = -\int \frac{\cos^m\left(\frac{\pi}{2}-x\right) d\left(\frac{\pi}{2}-x\right)}{\sin^n\left(\frac{\pi}{2}-x\right)}.$$

184. $$\int \frac{dx}{\sin^m x \cos^n x}$$
$$= \frac{1}{n-1}\cdot\frac{1}{\sin^{m-1} x \cdot \cos^{n-1} x} + \frac{m+n-2}{n-1}\int \frac{dx}{\sin^m x \cdot \cos^{n-2} x}$$
$$= -\frac{1}{m-1}\cdot\frac{1}{\sin^{m-1} x \cdot \cos^{n-1} x} + \frac{m+n-2}{m-1}\int \frac{dx}{\sin^{m-2} x \cdot \cos^n x}.$$

185. $$\int \frac{dx}{\sin^m x} = -\frac{1}{m-1}\cdot\frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1}\int \frac{dx}{\sin^{m-2} x}.$$

186. $$\int \frac{dx}{\cos^n x} = \frac{1}{n-1}\cdot\frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} x}.$$

187. $$\int \tan x\,dx = -\log\cos x.$$

188. $$\int \tan^2 x\,dx = \tan x - x.$$

189. $$\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx.$$

190. $$\int \operatorname{ctn} x\,dx = \log\sin x.$$

191. $$\int \operatorname{ctn}^2 x\,dx = -\operatorname{ctn} x - x.$$

192. $$\int \operatorname{ctn}^n x\,dx = -\frac{\operatorname{ctn}^{n-1} x}{n-1} - \int \operatorname{ctn}^{n-2} x\,dx.$$

193. $$\int \sec x\,dx = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$

194. $$\int \sec^2 x\,dx = \tan x.$$

195. $\int \sec^n x\,dx = \int \frac{dx}{\cos^n x}.$

196. $\int \csc x\,dx = \log \tan \tfrac{1}{2}x.$

197. $\int \csc^2 x\,dx = -\operatorname{ctn} x.$

198. $\int \csc^n x\,dx = \int \frac{dx}{\sin^n x}.$

199. $\int \frac{dx}{a+b\cos x} = \frac{-1}{\sqrt{a^2-b^2}} \cdot \sin^{-1}\left[\frac{b+a\cos x}{a+b\cos x}\right],$

or $\quad \frac{1}{\sqrt{b^2-a^2}} \log\left[\frac{b+a\cos x+\sqrt{b^2-a^2}\,.\,\sin x}{a+b\cos x}\right].$

200. $\int \frac{dx}{a+b\cos x+c\sin x}$

$$= \frac{-1}{\sqrt{a^2-b^2-c^2}} \cdot \sin^{-1}\left[\frac{b^2+c^2+a(b\cos x+c\sin x)}{\sqrt{b^2+c^2}\,(a+b\cos x+c\sin x)}\right]$$

$$= \frac{1}{\sqrt{b^2+c^2-a^2}} \cdot \log$$

$$\left[\frac{b^2+c^2+a(b\cos x+c\sin x)+\sqrt{b^2+c^2-a^2}\,(b\sin x-c\cos x)}{\sqrt{b^2+c^2}\,(a+b\cos x+c\sin x)}\right].$$

201. $\int x\sin x\,dx = \sin x - x\cos x.$

202. $\int x^2\sin x\,dx = 2x\sin x - (x^2-2)\cos x.$

203. $\int x^3\sin x\,dx = (3x^2-6)\sin x - (x^3-6x)\cos x.$

204. $\int x^m\sin x\,dx = -x^m\cos x + m\int x^{m-1}\cos x\,dx.$

205. $\int x\cos x\,dx = \cos x + x\sin x.$

206. $\int x^2\cos x\,dx = 2x\cos x + (x^2-2)\sin x.$

207. $\int x^3\cos x\,dx = (3x^2-6)\cos x + (x^3-6x)\sin x.$

208. $\int x^m \cos x\,dx = x^m \sin x - m\int x^{m-1}\sin x\,dx.$

209. $\int \frac{\sin x}{x^m}dx = -\frac{1}{m-1}\cdot\frac{\sin x}{x^{m-1}} + \frac{1}{m-1}\int\frac{\cos x}{x^{m-1}}dx.$

210. $\int \frac{\cos x}{x^m}dx = -\frac{1}{m-1}\cdot\frac{\cos x}{x^{m-1}} - \frac{1}{m-1}\int\frac{\sin x}{x^{m-1}}dx.$

211. $\int \frac{\sin x}{x}dx = x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \frac{x^9}{9\cdot 9!}\cdots.$

212. $\int \frac{\cos x}{x}dx = \log x - \frac{x^2}{2\cdot 2!} + \frac{x^4}{4\cdot 4!} - \frac{x^6}{6\cdot 6!} + \frac{x^8}{8\cdot 8!}\cdots.$

213. $\int \sin mx \sin nx\,dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}.$

214. $\int \cos mx \cos nx\,dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}.$

215. $\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2}.$

216. $\int \cos^{-1}x\,dx = x\cos^{-1}x - \sqrt{1-x^2}.$

217. $\int \tan^{-1}x\,dx = x\tan^{-1}x - \tfrac{1}{2}\log(1+x^2).$

218. $\int \operatorname{ctn}^{-1}x\,dx = x\operatorname{ctn}^{-1}x + \tfrac{1}{2}\log(1+x^2).$

219. $\int \operatorname{versin}^{-1}x\,dx = (x-1)\operatorname{versin}^{-1}x + \sqrt{2x-x^2}.$

220. $\int (\sin^{-1}x)^2\,dx = x(\sin^{-1}x)^2 - 2x + 2\sqrt{1-x^2}\sin^{-1}x.$

221. $\int x\cdot\sin^{-1}x\,dx = \tfrac{1}{4}[(2x^2-1)\sin^{-1}x + x\sqrt{1-x^2}].$

222. $\int x^n \sin^{-1}x\,dx = \frac{x^{n+1}\sin^{-1}x}{n+1} - \frac{1}{n+1}\int\frac{x^{n+1}dx}{\sqrt{1-x^2}}.$

223. $\int x^n \cos^{-1}x\,dx = \frac{x^{n+1}\cos^{-1}x}{n+1} + \frac{1}{n+1}\int\frac{x^{n+1}dx}{\sqrt{1-x^2}}.$

224. $\int x^n \tan^{-1}x\,dx = \frac{x^{n+1}\tan^{-1}x}{n+1} - \frac{1}{n+1}\int\frac{x^{n+1}dx}{1+x^2}.$

225. $\int \log x\, dx = x \log x - x.$

226. $\int \frac{(\log x)^n}{x} dx = \frac{1}{n+1} (\log x)^{n+1}.$

227. $\int \frac{dx}{x \log x} = \log . \log x.$

228. $\int \frac{dx}{x (\log x)^n} = - \frac{1}{(n-1)(\log x)^{n-1}}.$

229. $\int x^m \log x\, dx = x^{m+1} \left[\frac{\log x}{m+1} - \frac{1}{(m+1)^2} \right].$

230. $\int e^{ax} dx = \frac{e^{ax}}{a}.$

231. $\int x e^{ax} dx = \frac{e^{ax}}{a^2} (ax - 1).$

232. $\int x^m e^{ax} dx = \frac{x^m e^{ax}}{a} - \frac{m}{a} \int x^{m-1} e^{ax} dx.$

233. $\int \frac{e^{ax}}{x^m} dx = - \frac{1}{m-1} \frac{e^{ax}}{x^{m-1}} + \frac{a}{m-1} \int \frac{e^{ax}}{x^{m-1}} dx.$

234. $\int e^{ax} \log x\, dx = \frac{e^{ax} \log x}{a} - \frac{1}{a} \int \frac{e^{ax}}{x} dx.$

235. $\int e^{ax} \sin x\, dx = \frac{e^{ax} (a \sin x - \cos x)}{a^2 + 1}.$

236. $\int e^{ax} \cos x\, dx = \frac{e^{ax} (a \cos x + \sin x)}{a^2 + 1}.$

DEFINITE INTEGRALS.

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237. $\int_0^\infty \frac{a\,dx}{a^2+x^2} = \frac{\pi}{2}$, if $a>0$; 0, if $a=0$; $-\frac{\pi}{2}$, if $a<0$.

238. $\int_0^\infty x^{n-1}e^{-x}dx = \int_0^1 \left[\log\frac{1}{x}\right]^{n-1} dx = \Gamma(n)$.

$\Gamma(n+1) = n\cdot\Gamma(n)$. $\Gamma(2) = \Gamma(1) = 1$.

$\Gamma(n+1) = n!$, if n is an integer $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

239. $\int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_0^\infty \frac{x^{m+1}\,dx}{(1+x)^{m+n}} = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

240. $\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$

$= \frac{1.3.5\ldots(n-1)}{2.4.6\ldots(n)}\cdot\frac{\pi}{2}$, if n is an even integer.

$= \frac{2.4.6\ldots(n-1)}{1.3.5.7\ldots n}$, if n is an odd integer.

$= \frac{1}{2}\sqrt{\pi}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$, for any value of n.

241. $\int_0^\infty \frac{\sin mx\,dx}{x} = \frac{\pi}{2}$, if $m>0$; 0, if $m=0$; $-\frac{\pi}{2}$, if $m<0$.

242. $\int_0^\infty \frac{\sin x\,.\cos mx\,dx}{x} = 0$, if $m<-1$ or $m>1$;

$\frac{\pi}{4}$, if $m=-1$ or $m=1$; $\frac{\pi}{2}$, if $-1<m<1$.

243. $\int_0^\infty \frac{\sin^2 x\,dx}{x^2} = \frac{\pi}{2}$.

244. $\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

245. $\int_0^\infty \frac{\cos mx\,dx}{1+x^2} = \frac{\pi}{2}\cdot e^{-m}.$

246. $\int_0^\infty \frac{\cos x\,dx}{\sqrt{x}} = \int_0^\infty \frac{\sin x\,dx}{\sqrt{x}} = \sqrt{\frac{\pi}{2}}.$

247. $\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-k^2\sin^2 x}}$

$= \frac{\pi}{2}\left[1 + (\tfrac{1}{2})^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \left(\frac{1.3.5}{2.4.6}\right)^2 k^6 + \ldots\right]$, if $k^2 < 1$.

$= K.$

248. $\int^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2 x}\,.\,dx$

$= \frac{\pi}{2}\left[1 - (\tfrac{1}{2})^2 k^2 - \left(\frac{1.3}{2.4}\right)^2 \frac{k^4}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{k^6}{5} - \ldots\right]$, if $k^2 < 1$.

$= E.$

249. $\int_0^\infty e^{-a^2x^2}\,dx = \frac{1}{2a}\sqrt{\pi}\cdot = \frac{1}{2a}\Gamma(\tfrac{1}{2}).$

250. $\int_0^\infty x^n e^{-ax}\,dx = \frac{\Gamma(n+1)}{a^{n+1}} = \frac{n!}{a^{n+1}}.$

251. $\int_0^\infty x^{2n} e^{-ax^2}\,dx = \frac{1.3.5\ldots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}.$

252. $\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\,dx = \frac{e^{-2a}\sqrt{\pi}}{2}.$

253. $\int_0^\infty e^{-ax}\cos mx\,dx = \frac{a}{a^2+m^2}$, if $a > 0$.

254. $\int_0^\infty e^{-ax}\sin mx\,dx = \frac{m}{a^2+m^2}$, if $a > 0$.

255. $\int_0^\infty e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}\,.\,e^{-\frac{b^2}{4a^2}}}{2a}.$

256. $\int_0^1 \frac{\log x}{1-x}\,dx = -\frac{\pi^2}{6}.$

257. $\int_0^1 \frac{\log x}{1+x}\,dx = -\frac{\pi^2}{12}.$

258. $\int_0^1 \frac{\log x}{1-x^2} dx = -\frac{\pi^2}{8}.$

259. $\int_0^1 \log\left(\frac{1+x}{1-x}\right) \cdot \frac{dx}{x} = \frac{\pi^2}{4}.$

260. $\int_0^\infty \log\left(\frac{e^x+1}{e^x-1}\right) dx = \frac{\pi^2}{4}.$

261. $\int^1 \frac{dx}{\sqrt{\log\left(\frac{1}{x}\right)}} = \sqrt{\pi}.$

262. $\int_0^1 x^m \log\left(\frac{1}{x}\right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}.$

263. $\int_0^{\frac{\pi}{2}} \log \sin x\, dx = \int_0^{\frac{\pi}{2}} \log \cos x\, dx = -\frac{\pi}{2} \cdot \log 2.$

264. $\int_0^\pi x \cdot \log \sin x\, dx = -\frac{\pi^2}{2} \log 2.$

AUXILIARY FORMULAS.

The following formulas are sometimes useful in the reduction of integrals:

265. $\log u = \log cu +$ a constant.

266. $\log(-u) = \log u +$ a constant.

267. $\sin^{-1}u = \begin{cases} -\sin^{-1}\sqrt{1-u^2} & +\text{ a constant.} \\ -\frac{1}{2}\sin^{-1}(2u^2-1) & +\text{ a constant.} \\ \frac{1}{2}\sin^{-1}2u\sqrt{1-u^2} & +\text{ a constant.} \end{cases}$

268. $\tan^{-1}u = \begin{cases} -\tan^{-1}\dfrac{1}{u} & +\text{ a constant.} \\ \tan^{-1}\dfrac{u+c}{1-cu} & +\text{ a constant.} \end{cases}$

269. $\log(x \pm yi) = \frac{1}{2}\log(x^2+y^2) \pm i\tan^{-1}\dfrac{y}{x}.$

270. $\sin^{-1}u = \cos^{-1}\sqrt{1-u^2} = \tan^{-1}\dfrac{u}{\sqrt{1-u^2}} = \csc^{-1}\dfrac{1}{u}.$

271. $\cos^{-1}u = \sin^{-1}\sqrt{1-u^2} = \tan^{-1}\sqrt{\dfrac{1}{u^2}-1} = \sec^{-1}\dfrac{1}{u}.$

272. $\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\left(\dfrac{x \pm y}{1 \mp xy}\right).$

273. $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}).$

274. $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}(xy \mp \sqrt{(1-x^2)(1-y^2)}).$

275. $\sin x = \dfrac{e^{xi}-e^{-xi}}{2i}.$

276. $\cos x = \dfrac{e^{xi}+e^{-xi}}{2}.$

277. $\sin xi = \frac{1}{2}i(e^x - e^{-x}) = i\sinh x.$

278. $\cos xi = \frac{1}{2}(e^x + e^{-x}) = \cosh x.$

279. $\log_e x = (2.3025851)\log_{10}x.$

The Natural Logarithms of Numbers between 1.0 and 9.9.

N.	0	1	2	3	4	5	6	7	8	9
1.	0.000	0.095	0.182	0.262	0.336	0.405	0.470	0.531	0.588	0.642
2.	0.693	0.742	0.788	0.833	0.875	0.916	0.956	0.993	1.030	1.065
3.	1.099	1.131	1.163	1.194	1.224	1.253	1.281	1.308	1.335	1.361
4.	1.386	1.411	1.435	1.459	1.482	1.504	1.526	1.548	1.569	1.589
5.	1.609	1.629	1.649	1.668	1.686	1.705	1.723	1.740	1.758	1.775
6.	1.792	1.808	1.825	1.841	1.856	1.872	1.887	1.902	1.917	1.932
7.	1.946	1.960	1.974	1.988	2.001	2.015	2.028	2.041	2.054	2.067
8.	2.079	2.092	2.104	2.116	2.128	2.140	2.152	2.163	2.175	2.186
9.	2.197	2.208	2.219	2.230	2.241	2.251	2.262	2.272	2.282	2.293

The Natural Logarithms of Whole Numbers from 10 to 109.

N.	0	1	2	3	4	5	6	7	8	9
1	2.303	2.398	2.485	2.565	2.639	2.708	2.773	2.833	2.890	2.944
2	2.996	3.045	3.091	3.135	3.178	3.219	3.258	3.296	3.332	3.367
3	3.401	3.434	3.466	3.497	3.526	3.555	3.584	3.611	3.638	3.664
4	3.689	3.714	3.738	3.761	3.784	3.807	3.829	3.850	3.871	3.892
5	3.912	3.932	3.951	3.970	3.989	4.007	4.025	4.043	4.060	4.078
6	4.094	4.111	4.127	4.143	4.159	4.174	4.190	4.205	4.220	4.234
7	4.248	4.263	4.277	4.290	4.304	4.317	4.331	4.344	4.357	4.369
8	4.382	4.394	4.407	4.419	4.431	4.443	4.454	4.466	4.477	4.489
9	4.500	4.511	4.522	4.533	4.543	4.554	4.564	4.575	4.585	4.595
10	4.605	4.615	4.625	4.635	4.644	4.654	4.663	4.673	4.682	4.691

The Values in Circular Measure of Angles which are given in Degrees and Minutes.

1′	0.0003	9′	0.0026	3°	0.0524	20°	0.3491	100°	1.7453
2′	0.0006	10′	0.0029	4°	0.0698	30°	0.5236	110°	1.9199
3′	0.0009	20′	0.0058	5°	0.0873	40°	0.6981	120°	2.0944
4′	0.0012	30′	0.0087	6°	0.1047	50°	0.8727	130°	2.2689
5′	0.0015	40′	0.0116	7°	0.1222	60°	1.0472	140°	2.4435
6′	0.0017	50′	0.0145	8°	0.1396	70°	1.2217	150°	2.6180
7′	0.0020	1°	0.0175	9°	0.1571	80°	1.3963	160°	2.7925
8′	0.0023	2°	0.0349	10°	0.1745	90°	1.5708	170°	2.9671

NATURAL TRIGONOMETRIC FUNCTIONS.

Angle.	Sin.	Csc.	Tan.	Ctn.	Sec.	Cos.	
0°	0.000	∞	0.000	∞	1.000	1.000	90°
1	0.017	57.30	0.017	57.29	1.000	1.000	89
2	0.035	28.65	0.035	28.64	1.001	0.999	88
3	0.052	19.11	0.052	19.08	1.001	0.999	87
4	0.070	14.34	0.070	14.30	1.002	0.998	86
5°	0.087	11.47	0.087	11.43	1.004	0.996	85°
6	0.105	9.567	0.105	9.514	1.006	0.995	84
7	0.122	8.206	0.123	8.144	1.008	0.993	83
8	0.139	7.185	0.141	7.115	1.010	0.990	82
9	0.156	6.392	0.158	6.314	1.012	0.988	81
10°	0.174	5.759	0.176	5.671	1.015	0.985	80°
11	0.191	5.241	0.194	5.145	1.019	0.982	79
12	0.208	4.810	0.213	4.705	1.022	0.978	78
13	0.225	4.445	0.231	4.331	1.026	0.974	77
14	0.242	4.134	0.249	4.011	1.031	0.970	76
15°	0.259	3.864	0.268	3.732	1.035	0.966	75°
16	0.276	3.628	0.287	3.487	1.040	0.961	74
17	0.292	3.420	0.306	3.271	1.046	0.956	73
18	0.309	3.236	0.325	3.078	1.051	0.951	72
19	0.326	3.072	0.344	2.904	1.058	0.946	71
20°	0.342	2.924	0.364	2.747	1.064	0.940	70°
21	0.358	2.790	0.384	2.605	1.071	0.934	69
22	0.375	2.669	0.404	2.475	1.079	0.927	68
23	0.391	2.559	0.424	2.356	1.086	0.921	67
24	0.407	2.459	0.445	2.246	1.095	0.914	66
25°	0.423	2.366	0.466	2.145	1.103	0.906	65°
26	0.438	2.281	0.488	2.050	1.113	0.899	64
27	0.454	2.203	0.510	1.963	1.122	0.891	63
28	0.469	2.130	0.532	1.881	1.133	0.883	62
29	0.485	2.063	0.554	1.804	1.143	0.875	61
30°	0.500	2.000	0.577	1.732	1.155	0.866	60°
31	0.515	1.942	0.601	1.664	1.167	0.857	59
32	0.530	1.887	0.625	1.600	1.179	0.848	58
33	0.545	1.836	0.649	1.540	1.192	0.839	57
34	0.559	1.788	0.675	1.483	1.206	0.829	56
35°	0.574	1.743	0.700	1.428	1.221	0.819	55°
36	0.588	1.701	0.727	1.376	1.236	0.809	54
37	0.602	1.662	0.754	1.327	1.252	0.799	53
38	0.616	1.624	0.781	1.280	1.269	0.788	52
39	0.629	1.589	0.810	1.235	1.287	0.777	51
40°	0.643	1.556	0.839	1.192	1.305	0.766	50°
41	0.656	1.524	0.869	1.150	1.325	0.755	49
42	0.669	1.494	0.900	1.111	1.346	0.743	48
43	0.682	1.466	0.933	1.072	1.367	0.731	47
44	0.695	1.440	0.966	1.036	1.390	0.719	46
45°	0.707	1.414	1.000	1.000	1.414	0.707	45°
	Cos.	Sec.	Ctn.	Tan.	Csc.	Sin.	Angle.

Values of the Complete Elliptic Integrals, K and E, for Different Values of the Modulus, k.

$\sin^{-1}k$	K	E	$\sin^{-1}k$	K	E	$\sin^{-1}k$	K	E
0°	1.5708	1.5708	30°	1.6858	1.4675	60°	2.1565	1.2111
1°	1.5709	1.5707	31°	1.6941	1.4608	61°	2.1842	1.2015
2°	1.5713	1.5703	32°	1.7028	1.4539	62°	2.2132	1.1920
3°	1.5719	1.5697	33°	1.7119	1.4469	63°	2.2435	1.1826
4°	1.5727	1.5689	34°	1.7214	1.4397	64°	2.2754	1.1732
5°	1.5738	1.5678	35°	1.7312	1.4223	65°	2.3088	1.1638
6°	1.5711	1.5665	36°	1.7415	1.4248	66°	2.3439	1.1545
7°	1.5767	1.5649	37°	1.7522	1.4171	67°	2.3809	1.1453
8°	1.5785	1.5632	38°	1.7633	1.4092	68°	2.4198	1.1362
9°	1.5805	1.5611	39°	1.7748	1.4013	69°	2.4610	1.1272
10°	1.5828	1.5589	40°	1.7868	1.3931	70°	2.5046	1.1184
11°	1.5854	1.5564	41°	1.7992	1.3849	71°	2.5507	1.1096
12°	1.5882	1.5537	42°	1.8122	1.3765	72°	2.5998	1.1011
13°	1.5913	1.5507	43°	1.8256	1.3680	73°	2.6521	1.0927
14°	1.5946	1.5476	44°	1.8396	1.3594	74°	2.7081	1.0844
15°	1.5981	1.5442	45°	1.8541	1.3506	75°	2.7681	1.0764
16°	1.6020	1.5405	46°	1.8691	1.3418	76°	2.8327	1.0686
17°	1.6061	1.5367	47°	1.8848	1.3329	77°	2.9026	1.0611
18°	1.6105	1.5326	48°	1.9011	1.3238	78°	2.9786	1.0538
19°	1.6151	1.5283	49°	1.9180	1.3147	79°	3.0617	1.0468
20°	1.6200	1.5238	50°	1.9356	1.3055	80°	3.1534	1.0401
21°	1.6252	1.5191	51°	1.9539	1.2963	81°	3.2553	1.0338
22°	1.6307	1.5141	52°	1.9729	1.2870	82°	3.3699	1.0278
23°	1.6365	1.5090	53°	1.9927	1.2776	83°	3.5004	1.0223
24°	1.6426	1.5037	54°	2.0133	1.2681	84°	3.6519	1.0172
25°	1.6490	1.4981	55°	2.0347	1.2587	85°	3.8317	1.0127
26°	1.6557	1.4924	56°	2.0571	1.2492	86°	4.0528	1.0086
27°	1.6627	1.4864	57°	2.0804	1.2397	87°	4.3387	1.0053
28°	1.6701	1.4803	58°	2.1047	1.2301	88°	4.7427	1.0026
29°	1.6777	1.4740	59°	2.1300	1.2206	89°	5.4349	1.0008

The Common Logarithms of $\Gamma(n)$ for Values of n between 1 and 2.

n	$\log_{10}\Gamma(n)$	n	$\log_{10}\Gamma(n)$	n	$\log_{10}\Gamma(n)$	n	$\log_{10}\Gamma(n)$	n	$\log_{10}\Gamma(n)$
1.01	$\bar{1}.9975$	1.21	$\bar{1}.9617$	1.41	$\bar{1}.9478$	1.61	$\bar{1}.9517$	1.81	$\bar{1}.9704$
1.02	$\bar{1}.9951$	1.22	$\bar{1}.9605$	1.42	$\bar{1}.9476$	1.62	$\bar{1}.9523$	1.82	$\bar{1}.9717$
1.03	$\bar{1}.9928$	1.23	$\bar{1}.9594$	1.43	$\bar{1}.9475$	1.63	$\bar{1}.9529$	1.83	$\bar{1}.9730$
1.04	$\bar{1}.9905$	1.24	$\bar{1}.9583$	1.44	$\bar{1}.9473$	1.64	$\bar{1}.9536$	1.84	$\bar{1}.9743$
1.05	$\bar{1}.9883$	1.25	$\bar{1}.9573$	1.45	$\bar{1}.9473$	1.65	$\bar{1}.9543$	1.85	$\bar{1}.9757$
1.06	$\bar{1}.9862$	1.26	$\bar{1}.9564$	1.46	$\bar{1}.9472$	1.66	$\bar{1}.9550$	1.86	$\bar{1}.9771$
1.07	$\bar{1}.9841$	1.27	$\bar{1}.9554$	1.47	$\bar{1}.9473$	1.67	$\bar{1}.9558$	1.87	$\bar{1}.9786$
1.08	$\bar{1}.9821$	1.28	$\bar{1}.9546$	1.48	$\bar{1}.9473$	1.68	$\bar{1}.9566$	1.88	$\bar{1}.9800$
1.09	$\bar{1}.9802$	1.29	$\bar{1}.9538$	1.49	$\bar{1}.9474$	1.69	$\bar{1}.9575$	1.89	$\bar{1}.9815$
1.10	$\bar{1}.9783$	1.30	$\bar{1}.9530$	1.50	$\bar{1}.9475$	1.70	$\bar{1}.9584$	1.90	$\bar{1}.9831$
1.11	$\bar{1}.9765$	1.31	$\bar{1}.9523$	1.51	$\bar{1}.9477$	1.71	$\bar{1}.9593$	1.91	$\bar{1}.9846$
1.12	$\bar{1}.9748$	1.32	$\bar{1}.9516$	1.52	$\bar{1}.9479$	1.72	$\bar{1}.9603$	1.92	$\bar{1}.9862$
1.13	$\bar{1}.9731$	1.33	$\bar{1}.9510$	1.53	$\bar{1}.9482$	1.73	$\bar{1}.9613$	1.93	$\bar{1}.9878$
1.14	$\bar{1}.9715$	1.34	$\bar{1}.9505$	1.54	$\bar{1}.9485$	1.74	$\bar{1}.9623$	1.94	$\bar{1}.9895$
1.15	$\bar{1}.9699$	1.35	$\bar{1}.9500$	1.55	$\bar{1}.9488$	1.75	$\bar{1}.9633$	1.95	$\bar{1}.9912$
1.16	$\bar{1}.9684$	1.36	$\bar{1}.9495$	1.56	$\bar{1}.9492$	1.76	$\bar{1}.9644$	1.96	$\bar{1}.9929$
1.17	$\bar{1}.9669$	1.37	$\bar{1}.9491$	1.57	$\bar{1}.9496$	1.77	$\bar{1}.9656$	1.97	$\bar{1}.9946$
1.18	$\bar{1}.9655$	1.38	$\bar{1}.9487$	1.58	$\bar{1}.9501$	1.78	$\bar{1}.9667$	1.98	$\bar{1}.9964$
1.19	$\bar{1}.9642$	1.39	$\bar{1}.9483$	1.59	$\bar{1}.9506$	1.79	$\bar{1}.9679$	1.99	$\bar{1}.9982$
1.20	$\bar{1}.9629$	1.40	$\bar{1}.9481$	1.60	$\bar{1}.9511$	1.80	$\bar{1}.9691$	2.00	0.0000